MMU 6-8 Test Problems 21-25

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1 Problem 21

There are several possible outcomes of these three games. However, only four give us cases in which Mingchuan wins more games than David. We will analyze each case and calculate its probability.

1.1 Mingchuan wins 1, David wins 0

In this case, Mingchuan must have won 1 game and the other 2 must have ended in ties. The probability of this occurring is $\frac{1}{2} * \frac{1}{4} * \frac{1}{4} = \frac{1}{32}$. There are 3 possible games we can choose for Mingchuan to win, for a total probability of $\frac{3}{32}$

1.2 Mingchuan wins 2, David wins 0

In this case, Mingchuan must have won 2 gams and the other 1 must have ended in a tie. The probability of this occurring is $\frac{1}{2} * \frac{1}{2} * \frac{1}{4} = \frac{1}{16}$. There are 3 possible games we can choose for the tie, for a total probability of $\frac{3}{16}$

1.3 Mingchuan wins 3, David wins 0

In this case, Mingchuan must have won all 3 games. The probability of this occurring is $\frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$.

1.4 Mingchuan wins 2, David wins 1

The probability of this occurring is $\frac{1}{2} * \frac{1}{2} * \frac{1}{4} = \frac{1}{16}$. There are 3 possible games we can choose for David to win, for a total probability of $\frac{3}{16}$

So the total probability is $\frac{3}{32} + \frac{3}{16} + \frac{1}{8} + \frac{3}{16} = \frac{19}{32}$.

2 Problem 22

Consider first, the equilateral triangle with side lengths $10, 10, 10$. We know the area of an equilateral triangle of side length s is $s^2\sqrt{3}/4$. So the area of this triangle is $25\sqrt{3}$. Now, consider the next isoceles triangle, of side lengths $10, 13, 13$. Drawing an altitude from the tip to the base of length 10, and using the Pythagorean theorem, will show us that this triangle has height $\sqrt{13^2 - 5^2} = 12$. So, this triangle has area $\frac{1}{2} * 10 * 12 = 60$. Finally, we have the remainder of the circle outside both triangles. This sector has arc length $360 - 60 = 300$, as the equilateral triangle has angle 60. So, the area of this part of the circle is $\frac{300}{360} * 100\pi = \frac{250\pi}{3}$. So, the total area is $\frac{250\pi}{3} + 25\sqrt{3} + 60$. $a + b + c + d + e = 250 + 3 + 25 + 3 + 60 = 341$.

3 Problem 23

Let's first prime factorize N, such that we can analyze even and odd factors more effectively. This gives us $2^5 * 3 * 5 * 13 * 17 * 19^2 * 89$.The sum of all factors of N is then

$$(2^0+2^1+2^2+2^3+2^4+2^5)(3^0+3^1)(5^0+5^1)(13^0+13^1)(17^0+17^1)(19^0+19^1+19^2)(89^0+89^1)$$

.
To convince your self of this, try expanding this expression and see that each term is a factor of N. Now, just the odd factors of N will sum to

$$(3^0 + 3^1)(5^0 + 5^1)(13^0 + 13^1)(17^0 + 17^1)(19^0 + 19^1 + 19^2)(89^0 + 89^1)$$

.
So, the ratio of odd to total sum is

$$\frac{(3^0 + 3^1)(5^0 + 5^1)(13^0 + 13^1)(17^0 + 17^1)(19^0 + 19^1 + 19^2)(89^0 + 89^1)}{(2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5)(3^0 + 3^1)(5^0 + 5^1)(13^0 + 13^1)(17^0 + 17^1)(19^0 + 19^1 + 19^2)(89^0 + 89^1)} =$$

$$\frac{1}{2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5} = \frac{1}{63}.$$

Finally, the ratio of odd to even sum is $\frac{1}{62}$

4 Problem 24

Construct a good diagram of the hexagon, where F is the leftmost point and A is in the top-left. Now, imagine a coordinate system where point F is the origin. Then, point A is $(2, 2\sqrt{3})$ and B is $(6, 2\sqrt{3})$. Similarly, E is $(2, -2\sqrt{3})$ and D is $(6, -2\sqrt{3})$. Finally, C is $(8, 0)$. Then, point M will have coordinates $(7, \sqrt{3})$. We can then use Shoelace theorem on points F, M and D to get an area of $10\sqrt{3}$. So the answer is $10 + 3 = 13$

5 Problem 25

We can also express $\frac{1}{r^2}+\frac{1}{s^2}+\frac{1}{t^2}=(\frac{1}{r}+\frac{1}{s}+\frac{1}{t})^2-2(\frac{1}{rs}+\frac{1}{rt}+\frac{1}{st})$. First, $(\frac{1}{r}+\frac{1}{s}+\frac{1}{t})$ can be written as $\frac{st+rt+rs}{rst}$. By Vieta's formulas, this becomes $\frac{-21/18}{2/18}=\frac{21}{2}$. Next,$2(\frac{1}{rs}+\frac{1}{rt}+\frac{1}{st})$ can be written as $2(\frac{r+s+t}{rst})$. Again, by Vieta's, this is $2(\frac{25/18}{2/18})=25$. So, our final expression is $(\frac{21}{2})^2-25=\frac{341}{4}$