Spring Competition 2022 6-8 Answers and Solutions

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April 2022

Answers are on the next page, followed by solutions to each problem, one per page.

ANSWER KEY:

1. 55
2. 2
3. 7
4. 72
5. 404
6. 8
7. 160,000
8. 10
9. 3
10. 4
11. 20
12. 41
13. 10
14. 30
15. 4046
16. 10
17. 9
18. 10
19. 23
20. 6
21. 29
22. 21
23. 18
24. 1789
25. 101
26. 11
27. 11
28. 256
29. 32
30. 20

Problem 1. What is the sum of the first 10 positive integers?

We want to find $1+2+3+4+5+6+7+8+9+10$. Rearranging, this becomes $(1+10)+(2+9)+(3+8)+(4+7)+(5+6) = (11)+(11)+(11)+(11)+(11) = \boxed{55}$.

Problem 2. What is the slope of the line $4y - 8x = 2023$?

Writing this equation in standard form, we get $y = 2x + \frac{2023}{4}$. The slope of the line is the number in front of x, which is $\boxed{2}$.

Problem 3. Evaluate the following expression for $x = 2$ and $y = 3$:

$$\frac{x^3 + y^3}{x + y}.$$

The numerator evaluates to $2^3 + 3^3 = 8 + 27 = 35$. The denominator is $2 + 3 = 5$, so our answer is $\frac{35}{5} = \boxed{7}$.

Problem 4. Pat is buying a sandwich. He can choose between 3 types of bread, 4 types of meat, and 6 types of cheese. He must pick exactly 1 of each option. How many unique sandwiches can he buy?

Pat has 3 choices for the type of bread he wants, 4 choices for the type of meat, and 6 choices for the type of cheese. So, the total number of ways for him to make a sandwich is $3 \cdot 6 \cdot 4 = \boxed{72}$.

Problem 5. If $a \star b = a + b + ab$, and $x \star 4 = 2024$, what is the value of x?

Plugging in x and 4 into our definition of $\star$, we get $2024 = x + 4 + 4x$, which means $2020 = 5x$. This tells us $x = \boxed{404}$.

Problem 6. Lasya wants to travel to Hawaii for spring break, but she lives 4000 miles away. She flies to Hawaii in a plane moving at 500 miles per hour. How long does the flight last, in hours?

We will use the formula $d = rt$. The distance that she has to travel is 4000, and the rate is 500. So, $4000 = 500t$ which means $t = \boxed{8}$.

Problem 7. The numerical value of the perimeter of a square is exactly 1% of its numerical area. What is the area of the square?

Let the side length of the square be s. Then the perimeter of the square is $4s$, and the area is s^2. We are given that 100 times the perimeter is the area, so $100 \cdot (4s) = s^2$, meaning $s = 400$. So the area of the square is $400^2 = \boxed{160,000}$.

Problem 8. If 5 bingles equal 3 bungles, and 24 bungles equal 4 bengels, how many bingles are in a bengel?

If 24 bungles equals 4 bengels, 6 bungles is equal to 1 bengel. If 5 bingles equals 3 bungles, then 10 bingles equals 6 bungles.

Combining these two things, we get that 10 bingles is equal to 6 bungles which is equal to 1 bengal. So the answer is $\boxed{10}$.

Problem 9. What is the remainder when the 12 digit number 432143214321 is divided by 6?

By the divisibility trick for 3, our 12 digit number is divisible by 3, since the sum of the digits, 30 is. This tells us that the remainder when we divide the number by 6 is either 3 or 0.

Additionally, since the number is odd, it can't be divisible by 6. The only possibility remaining is that the remainder is $\boxed{3}$.

Problem 10. What is the sum of all solutions to the equation $|x-1|+|x-2|+|x-3|=5$?

If x is in between 1 and 2, let $x=1+t$, where t is between 0 and 1. Then $|x-1|=t$, $|x-2|=1-t$, and $|x-3|=2-t$. Thus $t+(1-t)+(2-t)=5$, which means $t=-2$, which doesn't work because it isn't in the range $(0,1)$.

Similarly, if x is between 2 and 3, there will be no solutions. The only other situations are if x is less than 1 or x is larger than 3.

If x is less than 1, then we must have $(1-x)+(2-x)+(3-x)=5$, which means $x=\frac{1}{3}$.

If x is larger than 3, then we must have $(x-1)+(x-2)+(x-3)=5$. The solution to this is $x=\frac{11}{3}$.

The sum of the values of x that work is $\frac{1}{3}+\frac{11}{3}=\boxed{4}$.

Problem 11. Gary has a mixture of 25% milk and 75% water, mixed thoroughly. He pours some of it out, and replaces all of that with milk. After doing that, he has exactly 60% water in his mixture. What percent of the total volume did he pour out at the beginning?

We must have poured out exactly 15 percent water in order to reduce it down to 60%. Since the mixture is homogeneous, removing 15 percent water means that we also have to remove $\frac{15}{3} = 5$ percent milk. In total, we have to remove $15 + 5 = \boxed{20}$ percent of the mixture at the start.

Problem 12. Mandy's 12 hour digital clock shows $11:20$ right now. In how many minutes will Mandy's clock show the digits $1, 1, 2, 0$ for the first time again, not necessarily in that order?

If there is a time containing 11 as the hour part, then the 2 and the 0 will be in the minutes slots. The only two possible times here are $11:20$ or $11:02$. Both of these are on or before our initial time, so this case doesn't work.

The next best thing is if 12 is the hour part. In that case, we must have 1 and 0 in minute part. The two times satisfying this are $12:01$ and $12:10$. $12:01$ is sooner than $12:10$, so $12:01$ is the first time that we have $1, 1, 2, 0$ in some order after our initial time. $12:01$ is $\boxed{41}$ minutes after $11:20$.

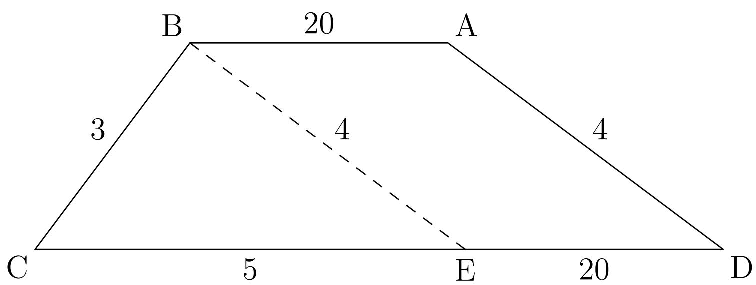
Problem 13. How many positive integers n less than 25 have the property that $\frac{(n-1)!}{n}$ is NOT an integer?

Notice that if n is a prime number, then there will be no factors of n in the numerator of the fraction. So, when n is a prime number, that fraction won't be an integer.

When n is a composite number, the fraction will usually be an integer, because $(n-1)!$ will generally contain a factor pair that multiplies to n. However, when $n = 4$, $3!$ does not have 2 factors of 2 in it, causing the fraction not to be an integer. All other composite numbers work.

The only other number that we have to check is 1, which works. So the only numbers that don't work are 4 and the prime numbers. The prime numbers less than 25 are $2, 3, 5, 7, 11, 13, 17, 19$, and 23. So there are a total of $\boxed{10}$ numbers that don't work.

Problem 14. $ABCD$ is an isosceles trapezoid with $AB||CD$. If $AB = 20$, $BC = 3$, $CD = 25$, and $AD = 4$, compute the area of triangle ACD.



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Translate segment AD over by 20 units. By doing this, A moves to point B, and D moves to point E such that $CE = 5$. Notice that BCE is a right triangle. The area of BCE is $\frac{3\cdot 4}{2} = 6$, so the height from B to CE must be $\frac{12}{5}$. So the area of triangle ACD is the height of the trapezoid, $\frac{12}{5}$, times the base, 25, divided by 2. This gives an answer of $\boxed{30}$.

Problem 15. What is the value of $2023.5^2 - 2022.5^2$?

Recall the formula $a^2 - b^2 = (a+b)(a-b)$. Plugging in 2023.5 and 2022.5 gives us $2023.5^2 - 2022.5^2 = (2023.5 + 2022.5)(2023.5 - 2022.5) = (4046)(1) = \boxed{4046}$.

Problem 16. How many positive multiples of 3 less than 10000 only use the digits 2 or 4?

In order for the number to be a multiple of 3, the sum of the digits of the number must be a multiple of 3. We will do casework based on how many digits the number has.

If the number has 1 digit, there is no way to make this work because neither 2 nor 4 is a multiple of 3.

If the number has 2 digits, the only possible numbers are 24 and 42.

If the number has 3 digits, the only possible things that work are 222 and 444.

If the number has 4 digits, we must have some permutation of 4422. There are $\frac{4!}{2!2!} = 6$ such permutations.

The total is $2 + 2 + 6 = \boxed{10}$ numbers that work.

Problem 17. What is the units digit of 9^{123}?

Listing out the units digit for the first few powers of 9, we notice the pattern $9, 1, 9, 1, 9, 1, 9, 1, \ldots$ repeats forever, with the odd powers giving a units digit of 9, and the even powers giving a units digit of 1. Since 123 is odd, 9^{123} ends in $\boxed{9}$.

Problem 18. Two different parabolas intersect at x points. Compute the sum of all possible values of x.

There are at most 4 intersection points between two parabolas. We will show that all values 4 and below actually can be values of x.

For 4 points, simply consider the parabolas $x = y^2 - 5$ and $y = x^2 - 5$.

For 3 points, we can take the two parabolas in the 4 case and shift the 'smiling' one up until it hits the bottom branch of the sideways parabola at exactly 1 point. There will be exactly 3 intersection points: 1 on the bottom branch and 2 on the top.

2 is easy: consider $y = x^2 - 1$ and $y = 1 - x^2$. 1 is also pretty simple, just consider $y = x^2$ and $y = -x^2$.

So the answer is $4 + 3 + 2 + 1 = \boxed{10}$.

Problem 19. a, b, and c are integers that satisfy $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 4$. What is the sum of all possible values of abc?

Notice that $\frac{1}{a} + \frac{2}{b}$ is at most $1 + 2 = 3$. This means that $\frac{3}{c} \geq 1$ and so $3 \geq c$. It is easy to see that negative values of c don't work. From here, we will do casework based on the value of c.

If $c = 1$, then $\frac{1}{a} + \frac{2}{b} = 1$. Expanding this gives $ab - 2a - b = 0$. This can be factored as $(a-1)(b-2) = 2$, which give us the solutions $(-1, 1, 1), (2, 4, 1)$, and $(3, 3, 1)$.

If $c = 2$, then $\frac{1}{a} + \frac{2}{b} = \frac{5}{2}$. Expanding again gives $5ab - 2b - 4a = 0$, which factors as $(5a-2)(5b-4) = 8$. This gives the solution $(2, 1, 2)$.

If $c = 3$, then $\frac{1}{a} + \frac{2}{b} = 3$. Clearly the only solution here is $(1, 1, 3)$, because of the inequality mentioned above.

The five values of abc corresponding to each solution are $-1, 8, 9, 4$, and 3. The sum is $\boxed{23}$.

Problem 20. Barry created a test that checks whether a person has an illness. The test is 95% accurate, and 1% of the population is infected with the illness. A person is chosen from the population at random and Barry's test is performed on them. What is the probability that the test says that that person has the illness? Express your answer to the nearest percent.

If the person has the illness, occurring with probability $\frac{1}{100}$, there is a $\frac{95}{100}$ chance that the test says that they have it. If the person doesn't have the illness, occurring with probability $\frac{99}{100}$, there is a $\frac{5}{100}$ chance that the test says that they have it. In total, the probability that the test says that the person has the illness is $\frac{95 \cdot 1 + 5 \cdot 99}{100^2} = 5.9\%$. Rounded to the nearest percent, this is $\boxed{6}$.

Problem 21. Beth writes down every integer from 1 to 111 in the following way. She writes down each digit of a number on a separate slip of paper, and then puts each slip in a bin. For example, when she writes 10, she will put 1 on one slip of paper, and then 0 on another slip. Then, Ben picks a random slip from her pile. The probability that the digit on Ben's slip is 1 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

We want to find the total number of times Beth wrote 1 divided by the total number of digits she wrote.

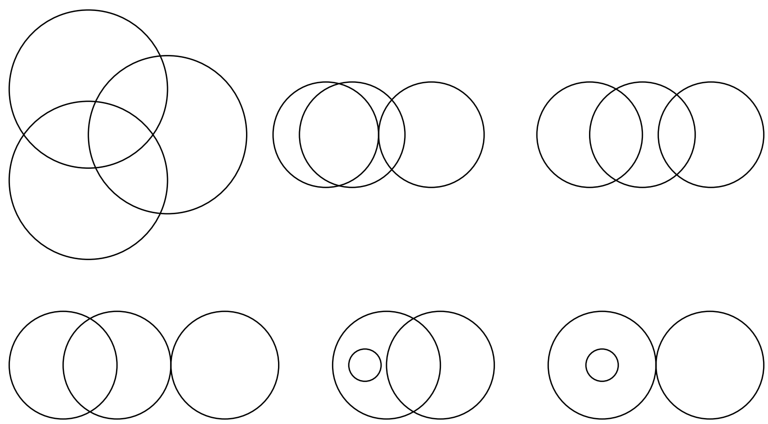
There are 3 possible places that a 1 could have appeared in a number: the hundreds digit, the tens digit, and the ones digit. There are 12 numbers with 1 as the hundreds digit, 12 numbers with 1 as the tens digit, and 12 more with 1 as the units digit. So the total number of times that Beth wrote 1 is 36.

To count the total number of digits Beth wrote in total, we can count the number of digits she wrote in each type of number. She wrote 9 one digit numbers, for a total of 9 digits written. She wrote 90 two digit numbers, for a total of 180 digits. She wrote 12 three digit numbers, for a total of 36 digits. Overall, she wrote $9 + 180 + 36 = 225$ digits.

So the probability we are looking for is $\frac{36}{225} = \frac{4}{25}$ which gives an answer of $\boxed{29}$.

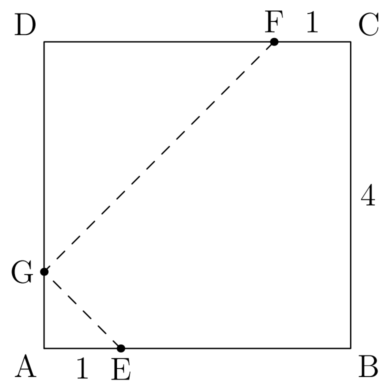
Problem 22. Three different circles are drawn in the plane. There are exactly S points that are on 2 or more of the circles. What is the sum of all possible values of S?

Two circles can intersect at most twice. So 3 circles can intersect at most 6 times. So the only possible values of S are $6, 5, 4, 3, 2$, or 1.



Demonstrations are shown for all 6 possible values above. This means that the answer is $6+5+4+3+2+1 = \boxed{21}$.

Problem 23. In square $ABCD$ with side length 4, points E and F are on sides AB and CD, respectively, such that $AE = CF = 1$. Point G is on side AD. What is the minimum possible value of $EG^2 + FG^2$?



Let $AG = r$. Then $DG = 4 - r$. By the Pythagorean theorem, $GE^2 = 1 + r^2$ and $FG^2 = 9 + (4 - r)^2 = 25 - 8r + r^2$. Then $FG^2 + EG^2 = 2r^2 - 8r + 26$. This is a parabola, with the minimum value occurring at $r = \frac{-b}{2a} = \frac{8}{2 \cdot 2} = 2$. Plugging in $r = 2$ gives $\min(FG^2 + EG^2) = \boxed{18}$.

Problem 24. There are exactly two positive integers $n < 2023$ such that the remainder when n^2 is divided by 2023 is the same as the remainder when $n + 369$ is divided by 2023. One of those integers is 235. Find the other one.

Let r be the other value that works. Then we have

$$235^2 \equiv 235 + 369 \pmod{2023}$$
$$r^2 \equiv r + 369 \pmod{2023}$$

Subtracting these two congruences gives us $235^2 - r^2 \equiv 235 - r \pmod{2023}$. By difference of squares, $235^2 - r^2 = (235 - r)(235 + r)$. Thus we have $(235 - r)(235 + r) - (235 - r) \equiv (235 - r)(234 + r) \equiv 0 \pmod{2023}$. Clearly, 235 and $2023 - 234 = 1789$ are solutions to this congruence. We are given that there are only 2 solutions, so the answer is $\boxed{1789}$.

Problem 25. a, b, and c are real numbers that satisfy the following three equations:

$$\begin{aligned} a + b &= 5 \\ abc &= 24 \\ b + c &= 10 \end{aligned}$$

The largest possible value of $a + c$ can be expressed as $\sqrt{m} + n$ for positive integers m and n. Find $m + n$.

From the first equation, $b = 5 - a$. Plugging this into the third equation gives $c = a + 5$. Plugging both of these into the middle equation gives $a(5 - a)(5 + a) = 24$. This simplifies to $a^3 - 25a + 24 = 0$.

That cubic has an obvious solution of $a = 1$, so we can factor out a $(a - 1)$ term. That gives us $(a - 1)(a^2 + a - 24) = 0$. Notice that the largest possible value of $a + c = 2a + 5$ will occur when a is as large as possible. The other two solutions for a are $\frac{-1+\sqrt{97}}{2}$ and $\frac{-1-\sqrt{97}}{2}$. Clearly $\frac{-1+\sqrt{97}}{2}$ is the largest out of the three solutions, giving the largest possible value of $a + c = 2a + 5$ to be $(-1 + \sqrt{97}) + 5 = 4 + \sqrt{97}$. So the answer is $\boxed{101}$.

Problem 26. For all positive integers n, let $S(n)$ denote the sum of the digits of n. For how many values of k less than 100 is $S(k) = S(2k)$?

Recall that $S(k) \equiv k \pmod 9$. Since $S(2k) = S(k)$, we must have $2k \equiv k \pmod 9$, which means k is a multiple of 9. It is easy to see that all multiples of 9 less than 100 work, including 99. So, there are a total of $\boxed{11}$ numbers that work.

Problem 27. r, s, and t are the roots of the cubic $x^3 - 7x - 1$. The value of $\frac{1}{1-r} + \frac{1}{1-s} + \frac{1}{1-t}$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Notice that $x^3 - 7x - 1$ must factor as $(x - r)(x - s)(x - t)$. Expanding tells us that $r + s + t = 0$, and $rs + st + rt = -7$. Additionally, $(1 - r)(1 - s)(1 - t) = (1)^3 - 7(1) - 1 = -7$.

Putting the given expression over a common denominator, we get

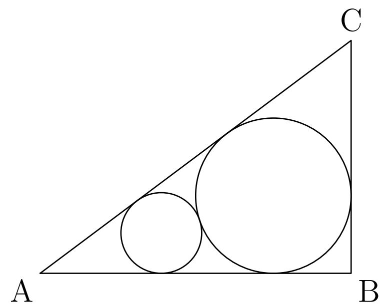
$$\frac{1}{1-r}+\frac{1}{1-s}+\frac{1}{1-t} = \frac{(1-r)(1-s)+(1-t)(1-s)+(1-r)(1-t)}{(1-r)(1-s)(1-t)} = \frac{3-2(r+s+t)+(rs+st+rt)}{(1-r)(1-s)(1-t)} = \frac{3-7}{-7} = \frac{4}{7}.$$

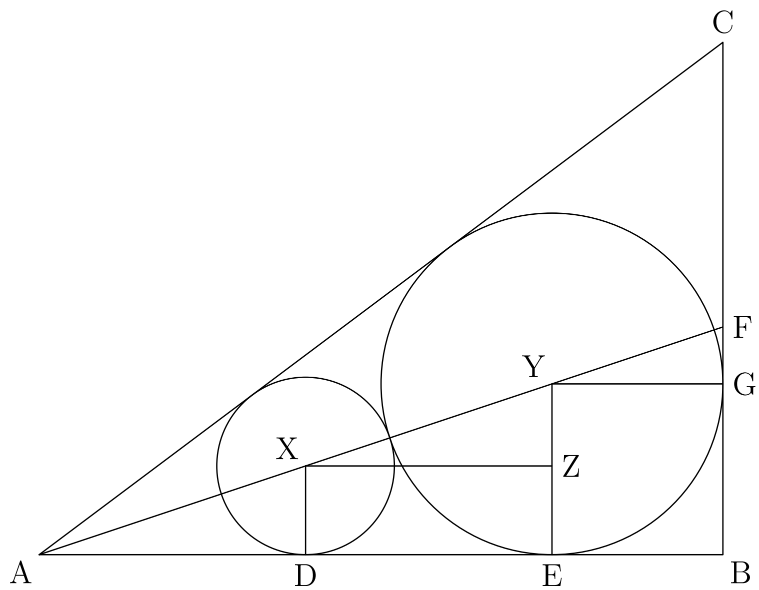
Thus the answer is $4 + 7 = \boxed{11}$.

Problem 28. Consider the 20 numbers $1^{20}, 2^{19}, 3^{18}, 4^{17}, \ldots, 19^2, 20^1$. Of these numbers, only one of them has more than 200 positive integer divisors. How many divisors does that number have?

The number with the largest number of factors is going to be a small number with a lot of prime divisors raised to a large number. 6 is an obvious first choice to test out. $6^{15} = 2^{15}3^{15}$, which has 256 factors. We are given that this is only value that has greater than 200 factors, so the answer is $\boxed{256}$.

Problem 29. A circle is inscribed in right triangle ABC, as shown. A smaller circle is tangent to sides AC and AB, as well as the larger circle. Given that the larger circle has radius 4, and the smaller circle has radius 1, compute the length of BC.





Drawing the radii from X and Y to AC, it becomes clear that X and Y both lie on the angle bisector of $\angle CAB$. From the given information, we have $XY = 5$ and $YZ = 3$. By the Pythagorean theorem, $XZ = 4$. Additionally, $EB = YG = 4$.

Additionally, since $AXD \sim XYZ$, we have $AD = \frac{4}{3}$. Thus $AB = \frac{4}{3} + 4 + 4 = \frac{28}{3}$. Since $ABF \sim XZY$, we have $FB = 7$. Let $FC = r$.

By the Pythagorean theorem, we have $AC = \sqrt{\left(\frac{28}{3}\right)^2 + (7+r)^2}$, and by the angle bisector theorem, $\frac{AC}{FC} = \frac{AB}{FB}$. This implies that $\frac{AC}{r} = \frac{4}{3}$.

We have $\sqrt{\left(\frac{28}{3}\right)^2 + (7+r)^2} = \frac{4r}{3}$, so $\left(\frac{28}{3}\right)^2 + (7+r)^2 = \frac{16r^2}{9}$. This is a quadratic with solutions $r = 25$ and $r = -7$, the latter of which is not okay. So $BC = r + 7 = \boxed{32}$.

Problem 30. Let $f(n)$ denote the number of diagonals in a regular polygon with n sides. Then the value of $\frac{1}{f(4)} + \frac{1}{f(5)} + \frac{1}{f(6)} + \frac{1}{f(7)} + \cdots$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Consider a regular n-gon. To pick a diagonal, we first have n choices for the first vertex, and then $(n-3)$ choices for the second one. However, we have to divide by 2 because every diagonal is being counted twice. So $f(n) = \frac{n(n-3)}{2}$, meaning $\frac{1}{f(n)} = \frac{2}{n(n-3)}$.

Notice that $\frac{2}{n(n-3)} = \frac{2}{3}\left(\frac{1}{n-3} - \frac{1}{n}\right)$. Decomposing a fraction like this is known as partial fraction decomposition. Plugging this back in, we get

$$\frac{1}{f(4)} + \frac{1}{f(5)} + \cdots = \frac{2}{3}\left(\left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right)\cdots\right).$$

Notice that every term in the form $\frac{1}{n}$, other than the first 3, is negative one time and positive another time, causing all of those terms to cancel out. Thus the sum is equal to $\frac{2}{3} \cdot \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right) = \frac{2}{3} \cdot \frac{11}{6} = \frac{11}{9}$, so the answer is $\boxed{20}$.